Mathematical Beauty behind Particle Production and Stokes Phenomena

At the James Scott Prize Lecture in 1939, P. A. M. Dirac emphasized the theory of functions of a complex variable as an interesting mathematical theory that fulfilled his criteria of beauty. He found this field to be of "exceptional beauty" and hence likely to lead to deep physical insight [1]. The lecture was delivered a decade after he discovered the Dirac equation, the positron was found, and the concept of the Dirac sea was well adopted.

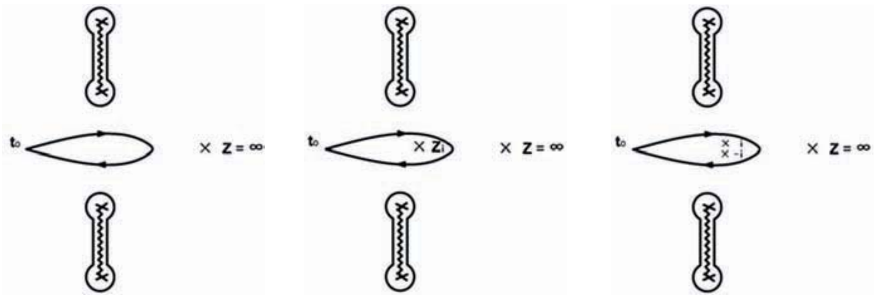

Fig. 1: Four contour integrals for the de Sitter radiation, in which simple poles at the north and south poles of a Euclidean geometry are responsible for the interference for Stokes phenomenon.

Dirac theory predicts that an external electric field may tilt the Dirac sea such that an electron in a negative energy can penetrate the mass gap via quantum tunneling into a positive energy electron and leave a positron (hole) as a real entity. This phenomenon is known as the Schwinger mechanism. Similarly, an expanding spacetime can produce particle pairs, one species of which is emitted to an observer, known as Hawking-Gibbons radiation, and the other species of which falls behind the horizon. Particle production from time-dependent backgrounds is one nonperturbative one-loop quantum effect.

S. P. Kim of Kunsan National University, Korea, has recently proposed that particle production is a consequence of the quantum evolution of particle path in the complex plane of time and further that the production rate is the magnitude of the sum of contour integrals over all possible independent paths in the complex plane [2, 3]:

$$N = \left| \sum_J exp\left(-i \oint_{C_J^{(1)}} \omega\,(z)\, dz\right) \right|$$

This simple formula is entirely characterized by the homotopy class of simple poles including a simple pole at the infinity of the frequency of a charged particle either in an electric field or an expanding spacetime, such as de Sitter space, and the sum exhausts all possible independent paths of the winding number 1. It depends neither on the actual locations of the simple poles nor does it require the information of actual paths in the complex plane. In fact, it is determined by the residues inside loops as well as the pole at the infinity. The formula realizes the mathematical beauty emphasized by Dirac, not to mention the wonderful nature of complex variables.

The formula results in the Stokes phenomenon in the global coordinates of a de Sitter space, which explains a peculiar dependence of particle production on the dimensionality of spacetime. Polyakov related this solitonic behavior to the Korteweg-de Vries equation and a reflectionless scattering Posch-Teller potential. In the phase-integral, Kim has shown that it is a consequence of the constructive or destructive interference between the actions along Stokes lines connecting two anti-Stokes lines [4]. Now the formula is the sum of four contour integrals as seen in Figure 1, and the production rate exhibits the Stokes phenomenon due to the interference of the north and the south pole of the de Sitter space [3].

Reference

[1] H. S. Kragh, DIRAC: A Scientific Biography (Cambridge University Press, 1990).
[2] S. P. Kim, "New geometric transition as origin of particle production in time-dependent backgrounds," Phys. Lett. B 725, 500 (2013).
[3] S. P. Kim, "Geometric origin of Stokes phenomenon for de Sitter radiation," Phys. Rev. D 88, 044027 (2013).
[4] S. P. Kim, "The Stokes phenomenon and quantum tunneling for de Sitter radiation in nonstationary coordinates," J. High Energy Phys. 09, 054 (2010).