Recent Progress In Class-$\mathcal{S}$ Construction and Non-Lagrangian Theories

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ABSTRACT

Four dimensional $\mathcal{N}$=2 superconformal field theories are extremely rich in dynamics, and yet, constrained heavily by superconformal symmetry algebra, leading to numerous fascinating dualities and applications. Among them are the strongly-coupled Argyres-Douglas theories without any Lagrangian description. They were previously believed to be engineered by the class-$\mathcal{S}$ construction involving irregular punctures, until very recently. Beem and Peelaers [1] proposed a novel class-$\mathcal{S}$ construction for some Argyres-Douglas theories using only regular ones. Here we briefly review this important progress and mention a few possible open problems.

4d $\mathcal{N}$=2 SCFTs and class $\mathcal{S}$ construction

4d $\mathcal{N}$=2 superconformal field theories (SCFTs) strike a fascinating balance between dazzling richness and mathematical rigidity (for pedagogical reviews, see e.g. [2, 3, 4, 5]). On the one hand, unlike the unique 4d $\mathcal{N}$=4 super-Yang-Mills, there are infinitely many 4d $\mathcal{N}$=2 SCFTs one can define by a range of different approaches, thus generating a vast and varying landscape for physicists to explore. On the other hand, they harbor a variety of rigid structures and exactly computable quantities that are highly constrained by the 4d $\mathcal{N}$=2 superconformal symmetry. Countless dualities originate from such richness and rigidity, such as the renowned Donaldson-Witten [6] and Seiberg-Witten invariants [7], the famous Alday-Gaiotto-Tachikawa duality (AGT duality) [8] between S^4-partition functions and correlation functions of Liouville/Toda theories on Riemann surfaces, and the SCFT/VOA (vertex operator algebra) correspondence [9]. These structures further relate to physics in other dimensions. For example, one can map [10, 11, 12] the correlation functions in the VOA [13] to the topological correlators [14] appearing in the 3d $\mathcal{N}$=4 SCFTs and their mirror dual [15, 16].

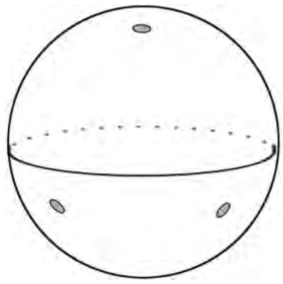

Fig. 1: A three punctured Riemann sphere. If the underlying $\mathfrak{j} = \mathfrak{su}(2)$ and all three punctures are the non-twisted regular punctures associated with trivial embedding, the resulting 4d effective theory is simply that of 4 free hypermultiplets. If $\mathfrak{j} = \mathfrak{su}(N)$ and all three punctures are associated with trivial embedding of $\Lambda : \mathfrak{su}(2) \to \mathfrak{su}(N >2)$, the corresponding 4d $\mathcal{N}$=2 SCFT is the well-known non-Lagrangian T_N theory.

Upon the wonderland of 4d $\mathcal{N}$=2 SCFTs, there are infinitely many point-like isolated kingdoms which are strongly coupled, and do not admit Lagrangian descriptions. Among them are the particularly interesting, yet simple, $T_N > 2$ theories [17, 18, 19] and Argyres-Douglas theories [20, 21]. Both of these theories admit "class-$\mathcal{S}$" construction [22, 23], i.e., compactifying some particular 6d theories on Riemann surfaces to engineer 4d effective theories. The T_N theories are constructed using only "regular punctures", while the latter always involve "irregular punctures", until very recently. A paper [1] pro-

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poses a particular construction of the Argyres-Douglas theories which only involve regular punctures of a special type. To further explain detail of their construction, let us first review the class S construction.

Classification of regular/irregular punctures

One way to construct 4d $\mathcal{N}=2$ SCFTs is to start from the famous and yet mysterious 6d (0,2) superconformal theories classified by a ADE algebra $\mathfrak{j}$, say, $\mathfrak{j} = \mathfrak{a}_n$, with the associated simply connected Lie group J. One can then put such a theory on a product manifold $\mathbb{R}^4 \times C_{g,n}$ where $C_{g,n}$ denotes a genus-g Riemann surface with n marked points. To consistently send the size of $C_{g,n}$ to zero and therefore engineer a 4d effective theory on $\mathbb{R}^4$ with the 4d $\mathcal{N}=2$ superconformal symmetry, one should first solve a system of partial differential equations of the 6d fields on the Riemann surface $C_{g,n}$, referred to as the Hitchin system. The 1-form Hitchin field in the equation is required to have prescribed singularities at those marked points: if the leading singularity is a simple pole, then the marked point is called a regular puncture, otherwise an irregular puncture. These punctures are further determined by additional data as follows.

- A non-twisted regular puncture is simplest and is further specified by its residue, which itself is defined by an embedding $\Lambda : \mathfrak{su}(2) \to \mathfrak{j}$. Recall that such an embedding is equivalent to a certain Young diagram.

- A less simple regular puncture, refereed to as a "twisted puncture", is defined by an outer-automorphism σ of $\mathfrak{j}$ (which then further specifies the invariant subalgebra $\mathfrak{j}_0$ of $\mathfrak{j}$), and an embedding $\Lambda : \mathfrak{su}(2) \to \mathfrak{j}_0^\vee$, the Langlands dual of the $\mathfrak{j}_0$. Table 1 lists all the allowed embedding for the case, $\mathfrak{j} = \mathfrak{a}_2$, corresponding to untwisted and twisted regular punctures.
- An irregular puncture $J^b[\kappa]$ is specified by two integers b, κ which control the order of the pole.

The number and detail of these punctures fix all the properties of the effective 4d SCFT. For example, they determine the (manifest) flavor symmetries and the associated flavor central charges, a,c central charges , the Coulomb branch spectrum, etc. In Figure[1], we display a three-punctured sphere. The detail of punctures then fixes the effective 4d theory to be 4 free hypermultiplets, or T_N theories, or other more complicated ones.

Argyres-Douglas theories

Argyres-Douglas theories are traditionally defined by $C = S^2$ with one irregular puncture, and optionally an additional regular puncture. They are strongly coupled, and have no deformation that preserves the 4d $\mathcal{N}=2$ superconformal symmetry. In particular, they have no Lagrangian description. The latter statement is manifested in the fact that they contain Coulomb branch chiral operators having fractional dimensions, which no 4d $\mathcal{N}=2$ Lagrangian theory possesses.

A set of extremely useful diagnostic tools are the superconformal index and its various limits. The most relevant ones in the paper we are describing are the Macdonald index and Schur index. Both of them are easy to compute via suitable topological quantum field theory on $C_{g,n}$. These quantities reveals the detailed structure of the operator algebra, including the representations in which operators transform, their quantum relations, and even their correlation functions.

Class $\mathcal{S}$ construction of AD theories with regular punctures

In Ref. [1], the authors consider a Riemann surface S^2 with three regular punctures, one untwisted, and a pair of twisted punctures. In particular, for the $\mathfrak{j} = \mathfrak{a}_2$ case, there are five possibilities, corresponding to different arrangements of the embeddings, as shown in Figure2.

Combining with Shapere-Tachikawa's central charge formula, they deduce the a,c anomaly coeffcients and the dimensions of the Coulomb branch chiral operators for the twisted $\mathfrak{a}_2$, and later twisted $\mathfrak{a}_{2n}$ theories. By analyzing their Macdonald index and subsequently the Schur and Hilbert series limit, they also gain insights into the operator contents and some quantum relations in the Higgs branch chiral ring. The associated VOAs are also constructed by working out the operator product expansions of the strong generators.

Table 1. Relevant embedding for $\mathfrak{j} = \mathfrak{a}_2$. Note that the second row correspond to twisted punctures, while the first row untwisted punctures. Twisted punctures must appear in pair. Here we use blue and green colors to denote embeddings for untwisted and twisted punctures respectively.

$\Lambda : \mathfrak{su}(2) \to a_2$	[1,1,1], [2,1]
$\Lambda : \mathfrak{su}(2) \to c_1$	[1,1], [2]

[1] The a,c central captures the conformal anomaly in four dimensions, given schematically by $T^\mu_\mu \sim c(\text{Weyl})^2 + a(\text{Euler})$. A flavor central charge associated to a flavor symmetry captures the leading singularity in the current-current OPE $j^a_\mu(x) j^b_\nu(y)$.

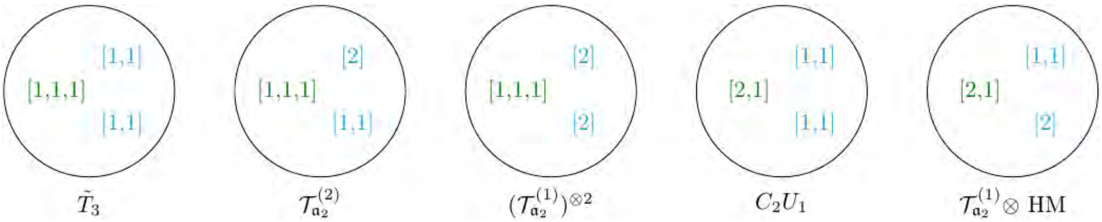

Fig. 2: Five twisted $\mathfrak{a}_2$ theories associated to different combinations of punctures. The third and the fifth theories will be most interesting, since they are related to Argyres-Douglas theories as discussed later in the main text. Again we use blue and green colors to denote untwisted and twisted punctures respectively.

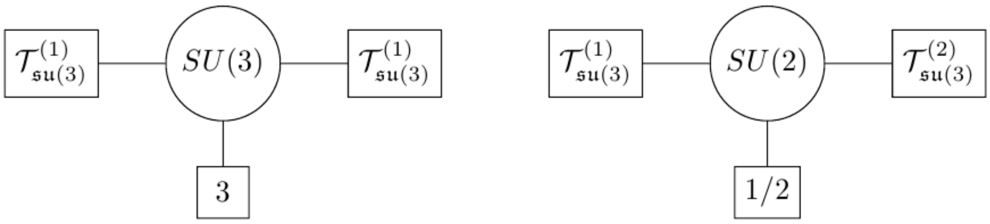

Fig. 3: Figures from Ref. [1] showing two S-dual theories made out of the twisted Argyres-Douglas theories $\mathcal{T}^{(i)}_{a_2}$.

Among the five possible twisted $\mathfrak{a}_2$ theories, two actually contain Coulomb branch operators of fractional dimensions! In particular, their Macdonald indices reveal that one of them (the last one in Figure 2) is actually the product of the well-studied (A_1, D_4) theory and a doublet of free hypermultiplets, while the other (the third one) is the product of two (A_1, D_4) theories. The former theory has $SU(3) \times SU(2)$ flavor symmetry where the $SU(2)$ acts on the free hypermultiplets, and the latter theory has $SU(3)^2$ symmetry which acts on the two tensor product ingredients individually. In both cases, the full flavor symmetry is enhanced from the manifest flavor symmetry visible from the punctures on the Riemann sphere. Furthermore, the five twisted $\mathfrak{a}_2$ theories are related by partial Higgsing.

Finally, the authors elaborate on some interesting S-dualities involving (A_1, D_4) theories. For example, take the product of two (A_1, D_4) theories. The diagonal $SU(3) \subset SU(3)^2$ of the product theory is gauged by an $SU(3)$ gauge group, and further coupled to a usual untwisted $\mathfrak{a}_2$ theory, aka 3^2 free hypermultiplets. S-duality then shows the equivalence of the two quiver theories shown in figure3, by performing pants decomposition of the punctured Riemann sphere in two different ways. On the right, the boxed 1/2 denotes a fundamental half-hypermultiplet, where the $SU(2) \subset SU(3)$ flavor of $\mathcal{T}^{(1)}_{a_2}$, and the $SU(2) \subset SU(3) \times SU(2)$ of the $\mathcal{T}^{(2)}_{a_2}$ are gauged by the central $SU(2)$.

There are open questions remaining to be explored. Detailed study of these twisted $\mathfrak{a}_{2\mathfrak{n}}$ theories with $n > 1$ are required to further understand the Coulomb branch spectrum. Also, 4d $\mathcal{N}=2$ theories admit a variety of non-local operators, while in the VOA language, surface operators are associated with non-vacuum modules [24]. Therefore, it would be interesting to study the module structure of these twisted theories to better understand the intricate dynamics brought about by non-local operators. Moreover, it is also natural to explore the 3d mirror dual of these theories and explicitly uncover the relation between the Higgs branch in the twisted Argyres-Douglas theories and the Coulomb branch correlation functions in the 3d mirror dual [10,11]. Finally, to further complete the AGT dictionary [8], it maybe important and plausible to find the corresponding vertex operators in the Liouville/Toda theory, and in turn define the "instanton partition function" for these twisted Argyres-Douglas theories, following the idea of an unpublished work by Nishinaka and collaborators.

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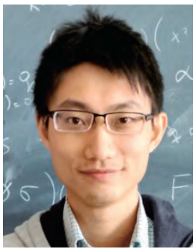

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